Order of Operations

Section 1: Introduction

You know from previous courses that if two quantities are added, it does not make a difference which quantity is added to which. For example, 5 + 6 = 6 + 5. This you will recall is the **commutative property of addition**. Even if you added more than two quantities it does not matter the order they are added. The same is true for multiplication. If you had to multiply $2 \cdot 4 \cdot 8$, you would always get 64 no matter what order the numbers were multiplied.

What happens though when the operations are not all only addition or all multiplication?

As an example, suppose you had to compute $2 + 5 \cdot 6$. You might perform the operations from left to right and get 42, whereas a friend of yours might first multiply the 5 and the 6 and then add 2, to get an answer of 32. In mathematics, we cannot have any inconsistencies such as this. It is important that any expression involving numbers and operations represent exactly one number. There are a set of rules (**order of operations**) to prevent different answers from occurring. Following the rules, the correct answer to the above example is 32. The order of operations has many names. For example, computer programmers usually use one of these terms: *order of precedence, math hierarchy*, or *order of computation.*

An understanding of the order of operations is fundamental in working with algebraic expressions like $5 + x^2$ and $(5 + x)^2$. These two expressions look similar, but the order of operations tells us they are interpreted very differently. The first expression, $5 + x^2$ means $5 + x \cdot x$, whereas $(5 + x)^2$ means $(5 + x)(5 + x)$.

When using a calculator to evaluate an expression such as $\frac{1.2 + 5.7}{0.3 \cdot 2.3}$, many people separately evaluate the **numerator** (top) and the **denominator** (bottom). They write these two answers down getting a final answer by dividing the two. If you know how to properly use grouping symbols and apply the order of operations, the answer can be obtained by entering a continuous succession of key strokes on a calculator. In some cases doing calculations separately can lead to rounding errors. As an example consider the following problem:

> If you travel from A to B at an average speed of 65 mph and return from B to A along the same route at an average speed of 55 mph, what is your average speed for the round trip?
>
> To solve this problem you can use the formula,
>
> $$\textbf{average speed} = \frac{2}{\frac{1}{r} + \frac{1}{s}}$$
>
> where r is the average speed from A to B and s is the average speed from B to A.

One of our goals in this unit, is to be able to perform the calculation $\dfrac{2}{\frac{1}{65}+\frac{1}{55}}$ on a calculator in a single series of keystrokes. If you do the calculation by first dividing 1 by 65, rounding that answer to three decimal places, then dividing 1 by 55, rounding to three decimal places, adding the two and then dividing 2 by that answer, you get an average speed of 60.61 mph. Doing the calculation in a single series of keystrokes gives an average speed of 59.58 mph, which is a significant difference. However, if you write the individual calculations with many decimal places you significantly reduce the rounding error.

In order to use a spreadsheet or program a computer in a language such as Java, it is essential to be able to write the above formula on a horizontal line which is the same way you would enter it on your calculator. It would look like 2/(1/65 + 1/55). Knowing the order of operations is essential to being able to write this expression correctly.

Section 2: Order of Operations

The following is a summary of the order of operations.

ORDER OF OPERATIONS

If **no** parentheses are present do the following:

1. Evaluate all powers, working from left to right.
2. Do any multiplications and divisions in the order in which they occur, reading from left to right.
3. Do any additions and subtractions in the order in which they occur, reading from left to right.

If parentheses are present, do the following:

1. First use the above steps within each pair of parentheses.
2. If the expression includes nested parentheses, evaluate the expression in the innermost set of parentheses first.

Caution: *Some people use the acronym "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" to remember the order of operations. Here the **P** stands for parentheses, **E** for exponent, **M** for multiplication, **D** for division, **A** for addition, and **S** for subtraction. Be careful when using this acronym. Many people interpret it to mean that multiplication must be done before division and that's not true. If you look at step 2 above, it says "in the order they occur from left to right." For example if you have* $24 \div 4 \cdot 2$*, you divide first, because it appears first working from left to right. So after dividing you would have* $6 \cdot 2 = 12$*. The same applies for addition and subtraction. You do whatever appears first. For example, if you have* $19 - 5 + 6$*,you subtract first because it appears first working from left to right. After subtracting you would have* 14 + 6 = 20. *Many people think you have to add first, but if do the answer would be* 19 - 11 = 8, *which is incorrect.*

Example 1 : Evaluate $7 + 3 \cdot 2 - 18 \div 6$ without using a calculator.

Solution : There are no parentheses or exponents, so do multiplications and divisions from left to right in the **order they appear**, then do additions and subtractions.

$$7 + 3 \cdot 2 - 18 \div 6 = 7 + \underset{3\cdot 2}{\underset{\uparrow}{6}} - \underset{18\div 6}{\underset{\uparrow}{3}}$$

$$= 13 - 3$$

$$= \mathbf{10}$$ ♦

Note: Often it is a good idea in complicated expressions to use extra parentheses to document the order of operations. For example, the expression $7 + 3 \cdot 2 - 18 \div 6$ can be written in a more readable form using parentheses: $7 + (3 \cdot 2) - (18 \div 6)$.

Example 2 : Evaluate $6 + 3(8 - 3)$ without using a calculator.

Solution : According to the order of operations, we have to calculate what is inside the parentheses first:

$$6 + 3(8 - 3) = 6 + 3(5)$$ *do what is inside the parentheses first*

$$= 6 + 15$$ *multiply before adding*

$$= \mathbf{21}$$ ♦

Example 3 : Evaluate $6 - 4[8 + 3(7 - 3)]$ without using a calculator.

Solution : This expression has nested parentheses where the [ ] are used in the same way as parentheses. In a case like this, we move to the innermost set of parentheses first and begin simplifying:

$$6 - 4[8 + 3(7 - 3)] = 6 - 4[8 + 3(4)]$$

$$= 6 - 4[8 + 12]$$

$$= 6 - 4[20]$$

$$= 6 - 80$$

$$= \mathbf{-74}$$ ♦

Example 4 : Evaluate $18 \div 3 \cdot 2 + 7 - 2 \cdot 4$ without using a calculator.

Solution : There are no parentheses or exponents, so we go from left to right and do all multiplications and divisions in the order they appear. In this case a division comes before a multiplication, so the division is done first.

$$\begin{aligned} 18 \div 3 \cdot 2 + 7 - 2 \cdot 4 &= 6 \cdot 2 + 7 - 2 \cdot 4 \\ &= 12 + 7 - 8 \\ &= 19 - 8 \\ &= \mathbf{11} \end{aligned}$$ ♦

Example 5 : Evaluate $9 \cdot 2^3 + 36 \div 3^2 - 8$ without using a calculator.

Solution : We do powers(exponents) first.

$$\begin{aligned} 9 \cdot 2^3 + 36 \div 3^2 - 8 &= 9 \cdot 8 + 36 \div 9 - 8 \\ &= 72 + 4 - 8 \\ &= 76 - 8 \\ &= \mathbf{68} \end{aligned}$$ ♦

Now do multiplications and divisions from left to right.

Example 6 : Evaluate $5 + 7(8 - 4)^2$ without using a calculator.

Solution: First, we must do what is inside the parentheses and then square that number. Next, we would multiply the answer by 7, then add 5.

$$5 + 7(8 - 4)^2 = 5 + 7(4)^2 = 5 + 7 \cdot 16 = 5 + 112 = 117$$ ♦

When computing an expression such as $\frac{2 \cdot 5 + 8}{6 - 3}$, the fraction bar has the same function as parentheses. Any operations that appear above or below a fraction bar should be completed first.

Example 7 : Compute $\frac{2\cdot 5 + 8}{6 - 3}$.

Solution: We have to compute 2· 5 + 8 and 6 - 3 before we divide.

$$\frac{2\cdot 5 + 8}{6 - 3} = \frac{10+8}{3} = \frac{18}{3} = \mathbf{6}$$

You should be able to write this problem on a horizontal line because, as was mentioned previously, this is how you enter it on a calculator or a computer.

So $\frac{2\cdot 5 + 8}{6 - 3}$ becomes (2· 5 + 8) ÷ (6 - 3) or (2· 5 + 8) / (6 - 3) ♦

Example 8 **:** Compute $0.5+\sqrt{\frac{9.2-0.2(6)}{2}}$

Solution: We have to compute what is under the square root first. Since the entire fraction is under the square root, we simplify that first, then take the square root.

$$0.5+\sqrt{\frac{9.2-0.2(6)}{2}} = 0.5+\sqrt{\frac{9.2-1.2}{2}} = 0.5+\sqrt{\frac{8}{2}} = 0.5+2 = 2.5$$

♦

Note: If you write this problem on a horizontal line to put in a calculator or computer, be sure to put parentheses around everything under the $\sqrt{\ }$ **.**

It would look like $0.5+\sqrt{((9.2-0.2(6))/2)}$ **.**

Activity 1

Evaluate the following **without** using a calculator:

1. $12 \div 6 \cdot 2 - 7(3 + 2)$

2. $4[6 - 4(2 - 7)]$

3. $5 \cdot 3^4 + 16 \div 8 - 2^2$

4. $\dfrac{8 \cdot 2 - 4}{10 - 4 \cdot 3}$

5. $7 - 3 + 2(9 - 6)^2$

6. $\frac{1}{5}[92 - \frac{12^2}{8}]$

It is important to be able to translate from mathematical symbols into words and vice versa. For example, 3 + 4(3 + 12), is translated into words as, *3 added to the product of 4 and the sum of 3 and 12*. Key words to remember: **sum** is the answer to an addition problem, **difference** is the answer to a subtraction problem, **product** is the answer to a multiplication problem, and **quotient** is the answer to a division problem.

Example 9 : Translate the following English expressions into mathematical expressions written in symbols.

a) 6 times the sum of 7 and 10
b) The sum of 3 times 2 and 12 times 5
c) Three times the difference of 14 and 8
d) 6 subtracted from the quotient of 24 and 8

Solution:

a) $6(7 + 10)$ → *Note that the words "sum of" required the use of ( ) so that we would add before multiplying.*

b) $3\cdot 2 + 12\cdot 5$ → *Here we do not need the ( ) because the order of operations tells us to multiply first, then add.*

c) $3(14 - 8)$ → *Note that the words "difference of" required the use of ( )so that we would subtract before multiplying.*

d) $24 \div 8 - 6$ or $\frac{24}{8} - 6$ → *We can use either a fraction bar or a division symbol, ÷, to indicate a quotient.*

◆

Activity 2

1. Translate the following English expressions into mathematical expressions written in symbols.

   a) 8 times the sum of 4 and 2.

   b) 4 added to 3 times the sum of 5 and 8

   c) The difference of 6 times 9 and the sum of 4 and 7.

2. Translate the following mathematical expressions into English expressions.

   a) $30 - (19 - 5)$

   b) $4\cdot 6 + 7$

   c) $(4 + 6) \div 8$

   d) $\frac{4\cdot 6}{6 - 9}$

Section 3: The Distributive Property

The expression 4(6 + 7) can be computed two ways:

1. using the order of operations, 4(6 + 7) = 4(13) = 52,

2. by using the **distributive property**.

The distributive property states that *the product of one number and the sum of two or more other numbers is equal to the sum of the products of the first number and each of the other numbers of the sum.* For example using the distributive property on the expression 4(6 + 7), we get 4(6 + 7) = 4· 6 + 4· 7 = 24 + 28 = 52.

The Distributive Properties
a(b + c) = ab + ac
a(b - c) = ab - ac

Terms and Factors

When an expression consists of several parts connected by addition and subtraction signs, each part (along with its sign) is known as a **term** of the expression. **Factors** are quantities that are multiplied together.

For example, in the expression 3 + 5· 6 the terms are 3 and 5· 6. The factors are 5 and 6. To properly apply the distributive property one has to distinguish between terms and factors. When you look at the distributive property you see that the parentheses are removed by multiplying each of the terms inside the parentheses by the other factor.

Example 10:

a) 5x and $3ab^2$ are the terms of the expression $5x + 3ab^2$.

b) $3x^2 y^5$ has only one term.

c) 2x - 7 has two terms, 2x and -7.

d) 2(x + 5) has only one term.

In the above example, $3x^2 y^5$ has only one term, however this one term it is made up of three **factors**. **Factors** are numbers (**coefficients**) or variables that are multiplied in a product. In this example the factors are 3, x^2 , and y^5 . **<u>In an expression, if you followed the order of operations and the last operation is multiplication, there is only one term.</u>**

Example 11 :

a) (x + 1)(x + 2) has two factors, (x + 1) and (x + 2).

b) 2x + 3 has one factor, 2x + 3.

♦

Activity 3

In each of the following algebraic expressions, name the terms and factors.

		number of terms	terms	factors
a)	2x	_____	_______	______
b)	$6x^2$	1	$6x^2$	6 and x^2
c)	6x + 1	_____	_______	______
d)	x(x + 1)	_____	_______	______
e)	(2 - x)(x + 1)	_____	_______	______
f)	4[6 - 4(2 - 7)]	_____	_______	______

Example 12 : Use the distributive property to simplify $5x+(1-x)(8)$.

Solution: Even though the 8 is to the right of the parentheses, we still distribute it. Then we would need to combine like terms to simplify.

$$\begin{aligned} 5x+(1-x)8 &= 5x+1(8)-(x)(8) \\ &= 5x+8-8x \qquad \text{combine } 5x-8x=-3x \\ &= -3x+8 \end{aligned}$$

♦

Now let's use the distributive property to evaluate some expressions.

Activity 4

Evaluate 3 + 5[6 - 9] first using the order of operations and then using the distributive property.

Evaluate 4[6 - 4(2 - 7)] using the distributive property.

Section 4: Using a Calculator

You have to remember that the minus sign is used in two different ways: (1) to indicate subtraction and (2) to designate a negative number. The [-] key is used for subtraction, and the [+/-] key is used to change the sign of a number. If you are using a **graphing calculator**, the negative sign of ⁻3.97 is entered with the [(-)] key before the 3.97 is entered. Also note that on a graphing calculator, you have a [ENTER] key rather than a [=] key.

Example 13 : Compute 5(17 + 53) using your calculator.

Solution: You want to enter the expression the way it is written using the parenthesis key.

[5] [x] [(] [17] [+] [53] [)] [=] 350 ♦

Example 14 : Compute $\frac{4+8}{2\cdot 3}$ using your calculator.

Solution: The calculator does not have a fraction bar, so we have to use the [÷] key for division. Translated into an English expression, $\frac{4+8}{2\cdot 3}$ says divide the sum of 4 and 8 by the product of 2 and 3. Note that the sum and product are <u>single quantities</u>, therefore they have to be enclosed in parentheses. The following would be entered in the calculator:

[(] [4] [+] [8] [)] [÷] [(] [2] [x] [3] [)] [=] 2 ♦

Activity 5

In the following perform the indicated operations on your calculator. Round answers to three decimal places.

a) 1.2(4.5 - 8.4)

b) -3.2(4.6 - ⁻ 5.9)

c) $\dfrac{123(40.2)}{24.65 - 12.93}$

Section 5: Evaluation of Formulas

Example 15 : The **circumference** (distance around) of a circle is given by the formula C = 2πr, where r is the radius of the circle. Find the circumference of a circle if the radius is 6.251 inches. Round your answer to 2 decimal places.

Solution: In this example we want to use the [π] key and not use an approximation like 3.14. Enter the following on your calculator.

[2] [x] [π] [x] [6.251] [=] **39.28 square inches (in^2)**

♦

Activity 6

Evaluation of formulas requires the substitution of numerical values for the letters (**variables**). In the expression 15y, 15 is called the **coefficient** of y and 15y means 15 times y. If $y = {}^{-}4$, then $15y = 15({}^{-}4) = {}^{-}60$.

Using your calculator evaluate $x(4x - 30)$, when $x = \frac{2}{3}$. First write down your key strokes.

Formulas involving subscripts

In most science courses, instead of using different letters to represent different quantities, the same letter is used with different **subscripts**. Just as a and b in an expression or formula represent different quantities, so do R_1 and R_2. Look at the nutritional value of a particular food and you will see vitamins such as Vitamin B_6 and Vitamin B_{12} identified by subscripts.

Activity 7

The equivalent resistance R (in ohms) of two resistors connected in parallel is given by the formula $R = \frac{R_1 R_2}{R_1 + R_2}$, where R_1 and R_2 are the resistances of the two resistors (in ohms). Find the value of R if
$R_1 = 15{,}000\ \omega$ and $R_2 = 3{,}000\ \omega$. Use your calculator.

Exercises for Order of Operations

Do all the exercises on separate paper, showing all work neatly.

1. Evaluate the following expressions with paper and pencil.
   Check your answers with a calculator.

   a) $3 \cdot 40 \div 12 \cdot 72 \div 24$ b) $3(45 - 21) + 7 - 10$

   c) $(45 - 21)(7 - 10)$ d) $64 - 5[14 + 20(14 - 3)]$

   e) $\dfrac{3(12 - 7) - 3}{4 + 8}$ f) $\dfrac{24 - 4(3 \cdot 5 - 2)}{2 \cdot 5} - 7$

   g) $\dfrac{11 - 5(3)}{7(3) - 10(-3)}$ h) $4(6 - \frac{3 - 6(2)}{3})$

   i) $123 - 2\{18 + 2(19 - 2[7 + 1])\} - 4$

2. Evaluate the following expressions, using $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

   a) $1.5a_1 + a_2 - 6a_3$ b) $a_3 - a_1 \cdot a_2$

   c) $\dfrac{5a_2 + 6a_1}{a_1 \cdot a_3 - 2}$ d) $5\{a_2 - a_1 [4 - (3 + a_3)] + 3\}$

   e) $a_1 a_3 - a_2(a_1 - a_3)$

3. Translate each of the following sentences into an expression using the operations +, -, $\cdot$, ÷ , and parentheses as needed. Then use your calculator to evaluate each expression.

   a) A car rental company charges $20 per day and 32.5 cents per mile for a compact car. The first 100 miles are free. If I rented the car for three days and drove 365 miles, how much will the rental company charge me?

   b) A transcript costs $3.00 for the first copy and $2.50 for each additional copy. If I want five transcripts sent, what is my total cost?

4. A friend of yours wanted to evaluate $\frac{1.2 + 5.7}{0.32 \cdot 2.39}$ on his calculator. He entered the following:

[(] [1.2] [+] [5.7] [)] [÷] [.32] [x] [2.39] [=]

a) Translate into an English expression the mathematical expression that was entered into the calculator.

b) Translate $\frac{1.2 + 5.7}{0.32 \cdot 2.39}$ into an English expression.

c) What is wrong with the way your friend entered the expression into the calculator?

5. On the previous problem, suppose another friend calculated the problem as follows:

1. add 1.2 and 5.7 to get 6.9

2. multiplied .32 by 2.39 and rounded this answer to .76

3. divided 6.9 by .76 to get 9.0789474.

Do you see anything wrong with doing the problem this way?

6. Name the terms in each of the following expressions.

a) $2\pi r$

b) $-\frac{1}{2}gt^2 + v_0 t + h_0$

ORDER OF OPERATIONS

Activity 1:

1)
$12 \div 6 \bullet 2 - 7(3+2)$
$2 \bullet 2 - 7(5)$
$4 - 35$
-31

2)
$4[6 - 4(2-7)]$
$4[6 - 4(-5)]$
$4 \bullet 26$
104

3)
$5 \bullet 3^4 + 16 \div 8 - 2^2$
$5 \bullet 81 + 16 \div 8 - 4$
$405 + 2 - 4$
$407 - 4$
403

4)
$\frac{8 \bullet 2 - 4}{10 - 4 \bullet 3}$
$\frac{16-4}{10-12}$
$\frac{12}{-2}$
-6

5)
$7 - 3 + 2(9-6)^2$
$7 - 3 + 2(3)^2$
$7 - 3 + 2(9)$
$7 - 3 + 18$
$4 + 18$
22

6)
$\frac{1}{5}[92 - \frac{(12^2)}{8}]$
$\frac{1}{5}[92 - \frac{144}{8}]$
$\frac{1}{5}[92 - 18]$
$\frac{1}{5}[74]$
14.8

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Activity 2:

1) a) $8(4+2)$ b) $4 + 3(5+8)$ c) $6 \bullet 9 - (4+7)$

2)
a) thirty minus the difference of nineteen and five
b) four times six, plus seven
c) the sum of four and six, divided by eight
d) the product of four and six, divided by the difference of six and nine

Activity 3:

expression	number of terms	terms	factors
a) 2x	1	2x	2 and x
b) $6x^2$	1	$6x^2$	6 and x^2
c) 6x + 1	2	6x and 1	(6 x + 1) only 1 factor
d) x(x + 1)	1	x(x + 1)	x and (x + 1)
e) (2 – x)(x + 1)	1	(2 – x)(x + 1)	(2 – x) and (x + 1)
f) 4[6-4(2-7)]	1	4[6-4(2-7)]	4 and [6-4(2-7)]

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Activity 4:

1) Evaluate 3 + 5[6 – 9]

a) using order of operations
3 + 5[-3]
3 – 15
-12

b) using the distributive property
3 + 5(6) – 5(9)
3 + 30 - 45
33 – 45
-12

2) Evaluate 4[6 – 4(2 – 7)]

a) using order of operations
4[6 – 4(-5)]
4[6 + 20]
4[26]
104

b) using the distributive property
4[6 – 4(2) –4(-7)]
4[6 – 8 + 28]
4(6) + 4(-8) + 4(28)
24 – 32 + 112
-8 + 112
104

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Activity 5:

a) 1.2 x (4.5 – 8.4) = -4.68

b) -3.2 x (4.6 - -5.9) = -33.6

c) (123 x 40.2) ÷ (24.65 – 12.93) = 421.894

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Activity 6:

Key strokes: [(] 2 / 3 [)] x [(] 4 * [(] 2 / 3 [)] – 30 [)] = -18.22

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Activity 7:

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$R = \frac{15000\,(3000)}{15000 + 3000} = \frac{45000000}{18000} = 2500 \ ohms$$